Waveguides Containing Moving Dispersive Media

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Abstract—**Perturbation formulas are derived for the changes in the dispersion curves and phase velocity for the modes in an arbitrary composite waveguide structure containing dispersive media in relative motion. The formulas are also valid when the media are fluids with arbitrary velocity distributions. It is shown that the relativistic transformation laws for the frequency and wave vector of uniform plane waves are also valid for waveguide modes provided that all moving media that make up the guide move with the same velocity. There are also difficulties when the moving media are dispersive. In general, one must therefore obtain the dispersion relation directly from the field equations or from the perturbation formulas. An example involving a simple surface wave along the interface of a moving plasma and a dielectric is worked out by both methods. As an interesting side result, it is found that plane waves in an unbounded isotropic plasma have phase velocities independent of the motion of the plasma.**

I. Introduction

When a plane wave propagates in a nondissipative nondispersive isotropic medium in the direction of motion of the medium, the frequency and propagation constant in the proper frame of the medium (Σ') are related to the corresponding quantities in the observer's frame (Σ) by the relativistic formulas

$$\omega' = \gamma(\omega - kv) \tag{1}$$

and

$$k' = \gamma(k - \beta\omega/c). \tag{2}$$

Here v is the velocity of the medium with respect to the observer's frame, $\beta = v/c$ and $\gamma = (1-v^2/c^2)^{-1/2}$. The frequency of the wave, as measured in the two frames, is given by ω and ω', respectively, and k and k' are the corresponding propagation constants.

These equations apply not only to uniform plane waves in a single medium, but with certain restrictions also to nonuniform waves in composite cylindrical structures where the field components vary typically as $\psi(x, y)e^{i(\omega t - kz)}$. All waveguide modes or surface waves fall into this category provided that the medium or the media that make up the guiding structure are nondispersive and move together with a single uniform velocity v (hereafter referred to as comoving waveguide structures). From these restrictions we can, of course, exclude any free-space portions of the guide cross section and any metal walls. The free-space portions can always be considered as moving with velocity v and conducting walls do not contain any fields. If the dispersion relations, i.e. the relation between k' and ω' is known in the proper frame, it can be computed in the observer's frame from (1) and (2).

The reason for the more general validity of (1) and (2) can be seen from their derivation which makes use only of the principle of the invariance of phase and the relativistic transformations of space and time. The equations clearly cannot apply to composite waveguides, containing two or more media (other than free space and conducting walls) moving with different velocities, nor to a single medium with a nonuniform velocity distribution. In this case there is no unique proper frame to which the equations could be applied.

When (1) and (2) cannot be applied, it is necessary to derive the dispersion relation in the observer's frame from first principles, starting from Maxwell's equations and the constitutive relations for moving media. In this paper, perturbation formulas are derived which give the change in the dispersion curve for any mode in a composite guide when the different dispersive media making up the guide are in relative motion. The perturbation formulas assume, of course, that the dispersion curve is known for the case where all media are at rest with respect to the observer.

The results are then applied to a simple example involving a TM surface wave along the interface of a semi-infinite moving plasma and a stationary dielectric. In this case, the field equations can be solved directly, thereby allowing a comparison with the perturbation theory results.

II. Addition Formulas for Phase Velocity

It will be necessary to review some well-known results for uniform plane waves in nondissipative isotropic media. These results are, however, stated in a form that is applicable also to nonuniform waves of the type indicated in comoving composite waveguide structures. The results are then specialized to plane waves in media of infinite extent and, in particular, to a moving isotropic plasma.

We shall restrict ourselves in this section to a single medium moving with a uniform velocity v or to a comoving composite waveguide structure as defined in the introduction. If we consider plane waves (uniform or nonuniform) propagating in the direction of v, the relation between the phase velocities measured in the two reference frames,

$$u = \omega/k,$$
$$u' = \omega'/k' \tag{3}$$

is easily obtained from (1) and (2). It is, in fact, the relativistic addition formula for velocities

$$u = \frac{u' + v}{1 + u'v/c^2}\,. \tag{4}$$

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In most practical problems of this type, v is small enough so that terms of order $(v/c)^2$ can be neglected. In that case, (4) becomes

$$u = u' + v[1 - (u'/c)^2]. \tag{5}$$

The last term is normally not negligible for fast waves for which u' is comparable with c. We know, however, that in certain types of composite waveguides and structures that support guided waves, slow waves can be excited for which $(u'/c)^2 \ll 1$. In that limit (5) becomes approximately

$$u \cong u' + v \tag{6}$$

and (1) and (2) may be replaced by

$$\omega' = \omega - kv \tag{7}$$

and

$$k' = k. \tag{8}$$

When we are dealing with uniform plane waves in an infinite medium, the phase velocity in the proper frame is given by

$$u' = c/n' = c/\sqrt{\kappa'\kappa_m'} \qquad n' = n(\omega') \tag{9}$$

where n' is the refraction index and κ' and κ_m' are the electric and the magnetic permittivities measured in the proper frame at the frequency ω'. If the medium is nondispersive, i.e., if n' is independent of the frequency, we can replace n' by n, the refractive index measured at frequency ω. In this case (5) reduces to the Fresnel relation

$$u = u_0 + v\left(1 - \frac{1}{n^2}\right) \tag{10}$$

where the proper phase velocity u' at frequency ω has been denoted by u_0. It is defined by

$$u_0 = c/n \qquad n \equiv n(\omega). \tag{11}$$

For a nondispersive medium we have, of course, $n' = n$ and $u' = u_0$.

For a uniform plane wave in a dispersive medium, the relation becomes a little more complex since we want to write the equation in terms of the refractive index n and the proper phase velocity u_0 at the frequency ω. Using a Taylor expansion we can write

$$n' \equiv n(\omega') = n(\omega - kv) = n - kv\frac{dn}{d\omega} = n - \frac{v\omega}{u}\frac{dn}{d\omega} \tag{12}$$

and

$$u' = u_0\left(1 + \frac{v\omega}{un}\frac{dn}{d\omega}\right) \tag{13}$$

provided $(v\omega/un)(dn/d\omega) \ll 1$. When this is substituted in (4) and terms of order $(v/c)^2$ are dropped, one obtains the well-known result

$$\begin{aligned} u &= u_0 + v\left(1 - \frac{1}{n^2} + \frac{\omega}{n}\frac{dn}{d\omega}\right) \\ &= u_0 + v\left(1 - \frac{1}{n^2} + \frac{\omega}{2n^2}\frac{dn^2}{d\omega}\right) \end{aligned} \tag{14}$$

which has been amply verified by experiment.[1]

If the plane wave moves in a direction making an angle α with the direction $\mathbf{v}$, it is not difficult to show that the last equation should be replaced by

$$u = u_0 + v\left(1 - \frac{1}{n^2} + \frac{\omega}{2n^2}\frac{dn^2}{d\omega}\right)\cos\alpha. \tag{15}$$

An isotropic plasma is a dispersive medium with

$$\begin{aligned} \kappa &= 1 - \frac{\omega_p^2}{\omega^2}, \\ \kappa_m &= 1, \\ n^2 &= \kappa \end{aligned} \tag{16}$$

if we neglect losses near the plasma resonance frequency ω_p. In a plasma of infinite extent, plane waves are possible only when the frequency is higher than the plasma frequency so that $\kappa > 0$. It is interesting to find that in this case the Fresnel drag coefficient $[1-(1/n^2)]$ is exactly cancelled by the dispersion term $(\omega/n)(dn/d\omega)$. This means that the phase velocity of plane waves in any inertial frame is independent of the magnitude and direction of the velocity of the plasma. We have simply

$$u = u_0 = c/\sqrt{\kappa}. \tag{17}$$

This is, in fact, true for arbitrary velocities as can be seen by substituting (16) with ω replaced by $\omega' = \gamma(\omega - kv)$, in (4) and solving for u.

Since waveguide modes can be synthesized from plane waves making some angle with the direction of propagation, the same result for the ordinary waveguide modes in guides filled with a moving plasma in view of (15) can be expected. The preceding results cannot be expected to hold in more complex guide structures, especially those containing two or more media in relative motion. Since measurements have recently been carried out to determine the drift velocity of a plasma from the experimental dispersion curves,[2] it is worthwhile to consider this problem in more detail.

III. Perturbation Theory for Waveguides Containing Moving Media

Let us consider a fairly general problem involving a composite waveguide structure. The total cross section S is subdivided into partial cross sections S_i $(i = 1, 2, 3, \cdots)$. Each of these partial sections may have a different dielectric constant and permeability, and may move with a different velocity parallel to the axis of the guide. Both dielectric constant and permeability may be functions of frequency. The velocity distribution need not be uniform in each section, so that fluid as well as rigid media are admissible. In this section two perturbation formulas are derived, one giving the perturbation in the propagation constant for a fixed frequency, the other the perturbation in the frequency for a fixed propagation constant.

Propagation Constant Perturbation

For a fixed frequency ω_0 the unperturbed and perturbed problems correspond to propagation constants k_0 and k, respectively. The fields will be assumed to vary as $e^{j(\omega_0 t - k_0 z)}$ and $e^{j(\omega_0 t - kz)}$ in the two cases. The unperturbed fields then satisfy

$$\nabla_t \times \boldsymbol{E}_0 - jk_0 \boldsymbol{e}_z \times \boldsymbol{E}_0 = -j\omega_0 \mu \boldsymbol{H}_0 \tag{18}$$

$$\nabla_t \times \boldsymbol{H}_0 - jk_0 \boldsymbol{e}_z \times \boldsymbol{H}_0 = j\omega_0 \epsilon \boldsymbol{E}_0 \tag{19}$$

while the perturbed fields satisfy the equations,[3] correct to the first order in the velocity

$$\nabla_t \times \boldsymbol{E} - jk\boldsymbol{e}_z \times \boldsymbol{E} - j\omega_0 \boldsymbol{\Lambda} \times \boldsymbol{E} = -j\omega_0 \mu' \boldsymbol{H} \tag{20}$$

and

$$\nabla_t \times \boldsymbol{H} - jk\boldsymbol{e}_z \times \boldsymbol{H} - j\omega_0 \boldsymbol{\Lambda} \times \boldsymbol{H} = j\omega_0 \epsilon' \boldsymbol{E} \tag{21}$$

$$\Delta k = -\omega_0 \frac{\int_S v\{4(\mu\epsilon - \mu_0\epsilon_0)N_0 + k_0[(\partial\epsilon/\partial\omega)\boldsymbol{E}_0 \cdot \boldsymbol{E}_0^* + (\partial\mu/\partial\omega)\boldsymbol{H}_0 \cdot \boldsymbol{H}_0^*]\}dS}{4\int_S N_0 dS}. \tag{29}$$

where $\boldsymbol{e}_z$ is a unit vector in the z direction and

$$\boldsymbol{\Lambda} = (\mu'\epsilon' - \mu_0\epsilon_0)\boldsymbol{v} = (\kappa'\kappa_m' - 1)\boldsymbol{v}/c^2. \tag{22}$$

The primes on ϵ and μ indicate that these quantities are to be given at the frequency ω' in the proper frame of the particular medium. If we use a Taylor expansion about the frequency ω_0 as in (12), we get

$$\nabla_t \times \boldsymbol{E} - jk\boldsymbol{e}_z \times \boldsymbol{E} - j\omega_0 \boldsymbol{\Lambda} \times \boldsymbol{E} = -j\omega(\mu - kv\partial\mu/\partial\omega)\boldsymbol{H} \tag{23}$$

and

$$\nabla_t \times \boldsymbol{H} - jk\boldsymbol{e}_z \times \boldsymbol{H} - j\omega_0 \boldsymbol{\Lambda} \times \boldsymbol{H} = j\omega(\epsilon - kv\partial\epsilon/\partial\omega)\boldsymbol{E}. \tag{24}$$

Since we are neglecting terms of order $(v/c)^2$, the $\boldsymbol{\Lambda}$ terms may be written with ϵ and μ replacing ϵ' and μ'.

When (18) and the complex conjugate of (24) are multiplied by $\boldsymbol{H}^*$ and $\boldsymbol{E}_0$, respectively, and the resultant expressions are subtracted, one obtains

$$\nabla_t \cdot (\boldsymbol{E}_0 \times \boldsymbol{H}^*) + j(k - k_0)\boldsymbol{e}_z \cdot \boldsymbol{E}_0 \times \boldsymbol{H}^* = -j\omega_0 \boldsymbol{\Lambda} \cdot \boldsymbol{E}_0 \times \boldsymbol{H}^* - j\omega_0 \mu \boldsymbol{H}_0 \cdot \boldsymbol{H}^* + j\omega_0(\epsilon - kv\partial\epsilon/\partial\omega)\boldsymbol{E}_0 \cdot \boldsymbol{E}^*. \tag{25}$$

Similarly, (19) and (23) lead to

$$\nabla_t \cdot (\boldsymbol{E}^* \times \boldsymbol{H}_0) + j(k - k_0)\boldsymbol{e}_z \cdot \boldsymbol{E}^* \times \boldsymbol{H}_0 = -j\omega_0 \boldsymbol{\Lambda} \cdot \boldsymbol{E}^* \times \boldsymbol{H}_0 + j\omega_0(\mu - kv\partial\mu/\partial\omega)\boldsymbol{H}_0 \cdot \boldsymbol{H}^* - j\omega_0 \epsilon \boldsymbol{E}_0 \cdot \boldsymbol{E}^*. \tag{26}$$

When these two expressions are added and integrated over the guide cross section, a number of terms cancel. The divergence terms can be changed to contour integrals which vanish on metallic boundaries for guides or at infinity for surface waves

$$(k - k_0)\int_S (\boldsymbol{E}_0 \times \boldsymbol{H}^* + \boldsymbol{E}^* \times \boldsymbol{H}_0) \cdot d\boldsymbol{S} = -\omega_0 \int_S (\boldsymbol{E}_0 \times \boldsymbol{H}^* + \boldsymbol{E}^* \times \boldsymbol{H}_0) \cdot \boldsymbol{\Lambda} dS - \omega_0 k \int v\left(\frac{\partial\epsilon}{\partial\omega}\boldsymbol{E}_0 \cdot \boldsymbol{E}^* + \frac{\partial\mu}{\partial\omega}\boldsymbol{H}_0 \cdot \boldsymbol{H}^*\right) dS. \tag{27}$$

We can now set the perturbed fields equal to the unperturbed fields, substitute for $\boldsymbol{\Lambda}$ and solve for $\Delta k = k - k_0$. If we also note that the component of the Poynting vector in the z direction is given by

$$N_0 = \tfrac{1}{2}\operatorname{Re}(\boldsymbol{e}_z \cdot \boldsymbol{E}_0 \times \boldsymbol{H}_0^*) = \tfrac{1}{4}(\boldsymbol{E}_0 \times \boldsymbol{H}^* + \boldsymbol{E}_0^* \times \boldsymbol{H}_0) \cdot \boldsymbol{e}_z \tag{28}$$

the final result can be written in the form (29).

Note that the integrations in the numerator have to be carried out only over those subsections S_i for which $v \neq 0$. Furthermore μ and ϵ may differ from section to section so that the numerator really consists of a sum of integrals.

If the waveguide contains only a single, nondispersive medium which has an arbitrary velocity distribution, we have

$$\Delta k = -\omega_0(\mu_1\epsilon_1 - \mu_0\epsilon_0)\frac{\int_S vN_0 dS}{\int_S N_0 dS}. \tag{30}$$

If all media are nondispersive and move with the same velocity, (29) reduces to

$$\Delta k = -\omega_0 v \frac{\int_S (\mu\epsilon - \mu_0\epsilon_0)N_0 dS}{\int_S N_0 dS}. \tag{31}$$

When there is only one nondispersive medium filling the entire cross section S and moving with uniform velocity, then[4]

$$\Delta k = -\omega_0 v(\mu\epsilon - \mu_0\epsilon_0) = -\omega_0 \Lambda.$$

Frequency Perturbation

If we want to find the perturbation in frequency for a fixed propagation constant k_0, we follow a somewhat similar procedure. The unperturbed equations, (18) and (19) remain unchanged, but the perturbed equations must be modified. We must replace k by k_0, and ω_0 by ω on the left-hand sides of (20) and (21). The quantities ϵ' and μ' on the right-hand side are now functions of $\omega'=\omega-k_0v$, rather than ω_0-kv. Hence, if $\Delta\omega=\omega-\omega_0$, $\epsilon=\epsilon(\omega_0)$ and $\mu=\mu(\omega_0)$ are defined

$$\nabla_t \times \boldsymbol{E} - jk_0\boldsymbol{e}_z \times \boldsymbol{E} - j\omega\boldsymbol{\Lambda} \times \boldsymbol{E} = -j\omega\{\mu + (\Delta\omega - kv)\partial\mu/\partial\omega\}\boldsymbol{H} \quad (32)$$

and

$$\nabla_t \times \boldsymbol{H} - jk_0\boldsymbol{e}_z \times \boldsymbol{H} - j\omega\boldsymbol{\Lambda} \times \boldsymbol{H} = j\omega\{\epsilon + (\Delta\omega - kv)\partial\epsilon/\partial\omega\}\boldsymbol{E}. \quad (33)$$

Following the same procedure as before, it is found that

$$\nabla_t\cdot(\boldsymbol{E}_0 \times \boldsymbol{H}^*) = -j\omega\boldsymbol{\Lambda}\cdot\boldsymbol{E}_0 \times \boldsymbol{H}^* - j\omega_0\mu\boldsymbol{H}_0\cdot\boldsymbol{H}^* + j\omega\{\epsilon + (\Delta\omega - kv)\partial\epsilon/\partial\omega\}\boldsymbol{E}_0\cdot\boldsymbol{E}^* \quad (34)$$

$$\nabla_t\cdot(\boldsymbol{E}^* \times \boldsymbol{H}_0) = -j\omega\boldsymbol{\Lambda}\cdot\boldsymbol{E}^* \times \boldsymbol{H}_0 - j\omega_0\epsilon\boldsymbol{E}_0\cdot\boldsymbol{E}^* + j\omega\{\mu + (\Delta\omega - kv)\partial\mu/\partial\omega\}\boldsymbol{H}_0\cdot\boldsymbol{H}^*. \quad (35)$$

Adding and integrating over the cross section, setting the perturbed fields equal to the unperturbed fields leads to

$$\frac{\Delta\omega}{\omega_0} = \frac{\int_S v\{4(\mu\epsilon - \mu_0\epsilon_0)N_0 + k_0[(\partial\epsilon/\partial\omega)\boldsymbol{E}_0\cdot\boldsymbol{E}_0^* + (\partial\mu/\partial\omega)\boldsymbol{H}_0\cdot\boldsymbol{H}_0^*]\}dS}{\int_S \{4U_0 + \omega_0[(\partial\epsilon/\partial\omega)\boldsymbol{E}_0\cdot\boldsymbol{E}_0^* + (\partial\mu/\partial\omega)\boldsymbol{H}_0\cdot\boldsymbol{H}_0^*]\}dS} \quad (36)$$

where U_0 is the volume density of field energy given by

$$U_0 = \tfrac{1}{4}(\epsilon\boldsymbol{E}_0\cdot\boldsymbol{E}_0^* + \mu\boldsymbol{H}_0\cdot\boldsymbol{H}_0^*). \quad (37)$$

If all media are nondispersive and move with the same velocity, (36) reduces to

$$\frac{\Delta\omega}{\omega_0} = \frac{v\int_S (\mu\epsilon - \mu_0\epsilon_0)N_0 dS}{\int_S U_0 dS}. \quad (38)$$

It is not too difficult to show that (30) and (38) are consistent with (1) and (2). This can be demonstrated in connection with Fig. 1 which shows linearized sections of the dispersion curves for $v=0$ and for $v\neq 0$. For small v/c, the points A, B, and C, obtained from (30), (38), and (1) and (2), respectively, must lie on the same straight line. From the figure, the condition can clearly be written in the form

$$\frac{k_0v - \Delta\omega}{\omega_0 v/c^2} = -\frac{\Delta\omega}{\Delta k}. \quad (39)$$

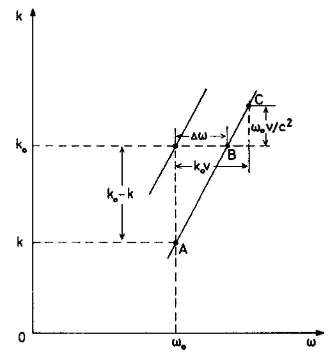

Fig. 1. Perturbed and unperturbed dispersion curves.

If we substitute from (30) and (38), we find that (39) is satisfied provided

$$\frac{\int_S \mu\epsilon N_0 dS}{\int U_0 dS} = \frac{k_0}{\omega_0} = \frac{1}{u_0}. \quad (40)$$

That this is generally true for a composite waveguide can be shown from (18) and (19). (See Appendix.)

IV. Slow Waves along the Interface of a Semi-Infinite Plasma

As an example of the application of the formulas of Section IV we shall choose the problem of a surface wave along the interface of a moving semi-infinite plasma and a dielectric. This problem has some bearing on recent experiments.[2]

It is well known[5] that along the interface of a semi-infinite plasma and free space, a slow surface wave can propagate which decays exponentially away from the interface. If the plasma has a drift velocity v parallel to the interface, the solution is unaltered in the proper frame of the plasma. The solution in the observer's frame can then be obtained simply by a Lorentz transformation.

When the free space above the plasma is replaced by a dielectric which is not moving with the plasma, the situation is not quite as simple. We must use the constitutive relations for a moving medium whether we work in the proper frame of the plasma or that of the dielectric. Let us choose the proper frame of the dielectric as our reference frame. The interface is taken as the z-y plane and the plasma drifts in the z direction with velocity v (see Fig. 2). Subscripts 1 and 2

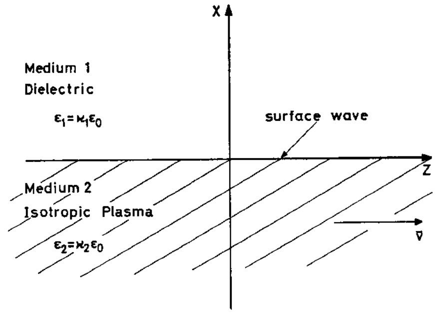

Fig. 2. Geometry of problem.

will refer to the dielectric and plasma region, respectively. We shall first obtain a direct solution of the field equations and then compare the results with those obtained from the perturbation formulas.

In region 2, we must supplement the source-free Maxwell equations by the first-order constitutive relations[3] (terms of v^2/c^2 or Λ^2 are neglected here and later on):

$$\boldsymbol{D}_2 = \epsilon_2 \boldsymbol{E}_2 + \boldsymbol{\Lambda} \times \boldsymbol{H}_2,$$
$$\boldsymbol{B}_2 = \mu_0 \boldsymbol{H}_2 - \boldsymbol{\Lambda} \times \boldsymbol{E}_2. \tag{41}$$

If fields vectors are assumed proportional to

$$e^{\alpha_2 x + j(\omega t - kz)} \tag{42}$$

Maxwell's equations separate into two independent sets, corresponding to a TM and a TE wave, respectively,

$$(k + \Lambda\omega)H_{2y} = \omega\epsilon_2 E_{2x} \qquad (k + \Lambda\omega)E_{2y} = -\omega\mu_0 H_{2x}$$
$$-j\alpha_2 H_{2y} = \omega\epsilon_2 E_{2z} \qquad -j\alpha_2 E_{2y} = -\omega\mu_0 H_{2z}$$
$$(k + \Lambda\omega)E_{2x} - j\alpha_2 E_{2z} = \omega\mu_0 H_{2y} \qquad (k + \Lambda\omega)H_{2x} - j\alpha_2 H_{2z} = -\omega\epsilon_2 E_{2y} \tag{43}$$

Both sets will have a nontrivial solution only if

$$(k + \Lambda\omega)^2 = \alpha_2^2 + \kappa_2(\omega/c)^2 \qquad \kappa_2 = \epsilon_2/\epsilon_0. \tag{44}$$

In the dielectric, we will assume fields of the form

$$e^{-\alpha_1 x + j(\omega t - kz)}. \tag{45}$$

We get a set of equations of the form of (43) except that the subscripts on the fields are now 1 instead of 2 and α_2 is replaced by $-\alpha_1$. We have a nontrivial solution if

$$k^2 = \alpha_1^2 + \kappa_1(\omega/c)^2 \qquad \kappa_1 = \epsilon_1/\epsilon_0. \tag{46}$$

The continuity requirements on the tangential components of $\boldsymbol{E}$ and $\boldsymbol{H}$ lead to the relations[5]

$$\alpha_2 = -(\kappa_2/\kappa_1)\alpha_1 \quad \text{for the TM wave} \tag{47}$$

and

$$\alpha_2 = -(\mu_2/\mu_1)\alpha_1 \quad \text{for the TE wave.} \tag{48}$$

These equations can clearly not be satisfied unless one of the dielectric constants or one of the permeabilities is negative. For a plasma, κ_2 will be negative for frequencies below the plasma frequency, so that a slow TM surface wave is possible. For ferrites a TE wave may be possible in certain frequency ranges.

For our drifting plasma, we can combine (44), (46), (22), and (47) to obtain the dispersion relation

$$k^2(\kappa_2^2 - \kappa_1^2) - (2k\omega v/c^2)\kappa_1^2(\kappa_2 - 1) - (\omega/c)^2\kappa_1\kappa_2(\kappa_2 - \kappa_1) = 0. \tag{49}$$

It will be convenient in what follows to introduce the dimensionless variables defined by

$$\beta = v/c,$$
$$X = \omega/\omega_p,$$
$$Y = kc/\omega_p, \tag{50}$$

In terms of these symbols, (49) can be written in the form

$$(\kappa_2^2 - \kappa_1^2)Y^2 - 2\beta XY\kappa_1^2(\kappa_2 - 1) - X^2\kappa_1\kappa_2(\kappa_2 - \kappa_1) = 0. \tag{51}$$

The dielectric constant of the plasma is given by (16), but it must be evaluated at the frequency $\omega' = \omega - kv$. Hence

$$\kappa_2 = 1 - (X - \beta Y)^{-2}. \tag{52}$$

For $v = 0$ ($\beta = 0$) we get the well-known solution[5]

$$(kc/\omega_p)^2 = Y_0^2 = X_0^2 \frac{\kappa_1\kappa_2}{\kappa_2 + \kappa_1} = X_0^2 \frac{\kappa_1(1 - X_0^2)}{1 - (1 + \kappa_1)X_0^2}. \tag{53}$$

A real solution is obtained only at frequencies below the cutoff frequency given by

$$X_{0c} = (1 + \kappa_1)^{-1/2}$$

or

$$\omega_c/\omega_p = (1 + \kappa_1)^{-1/2}. \tag{54}$$

We are interested in how the dispersion curve in (53) is changed when β has a finite positive or negative value. This is most conveniently expressed in terms of the shift of the dispersion curve ΔX in the X-Y plane for a given Y. If we substitute (52) in (51), multiply through by $(X - \beta Y)^4$ and then neglect terms of order β^2 and higher, the dispersion relation becomes

$$F(X, Y, \beta) \equiv Y^2[1 - 2X^2 - (\kappa_1^2 - 1)X^4] - \kappa_1 X^2[1 - (2 - \kappa_1)X^2 - (\kappa_1 - 1)X^4] + 4\beta XY\{Y^2[1 + (\kappa_1^2 - 1)X^2] + X^2(1 - X^2)\kappa_1(\kappa_1 - 1)\} = 0. \tag{55}$$

Equation (53) gives a zero-order solution to (55). The first-order change in X for a fixed Y is given by Newton's rule

$$\Delta X = -\frac{F(X, Y, \beta)}{\partial F/\partial X} \tag{56}$$

with

$$Y = Y_0.$$

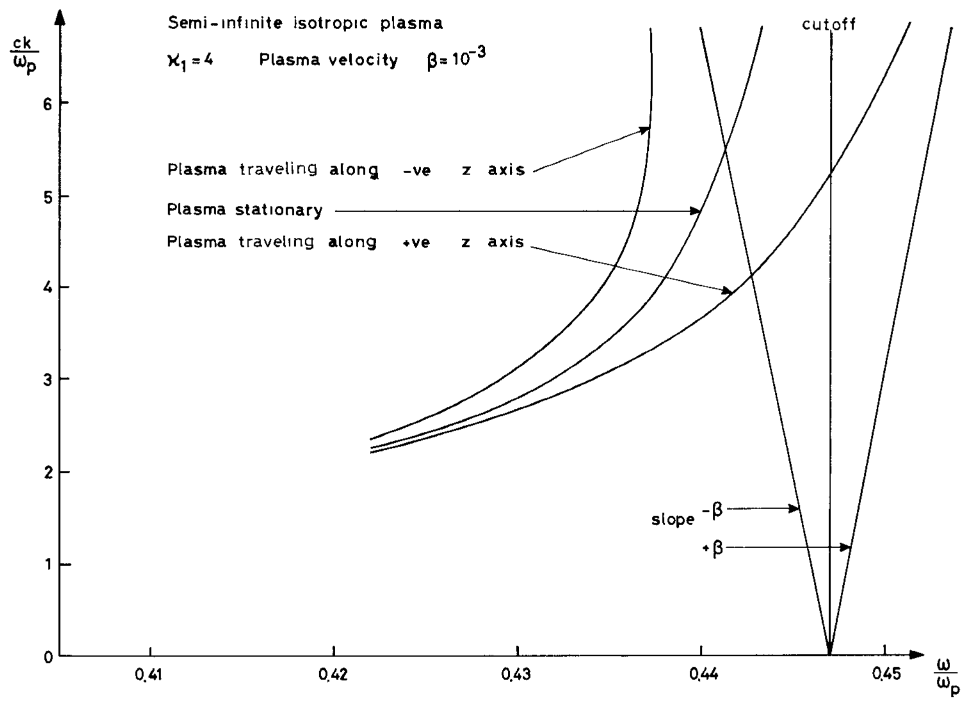

Fig. 3. Dispersion curves for TM surface waves on semi-infinite isotropic plasma.

After a considerable amount of algebra, dropping terms of order β^2, we get

$$\frac{\Delta X}{\beta Y_0} = \frac{\Delta\omega}{k_0 v} = \frac{(1+\kappa_1)X_0^2(1-X_0^2)}{1-2X_0^2+(\kappa_1+1)X_0^4}. \tag{57}$$

This equation gives the change in ω required to maintain the same propagation constant, when the drift velocity of the plasma changes from zero to v.

The ratio given by (57) tends to unity as X approaches the cutoff value X_c or, equivalently, as ω tends to ω_c. This means that the $\omega - k$ curve has the asymptote $\omega = \omega_c + kv$. This is shown for zero, positive and negative values of v in Fig. 3 for $\kappa_1 = 4$.

Equation (37) is plotted in Fig. 4 with κ_1 as a parameter. The frequency is normalized with respect to the cutoff frequency ω_c. It is seen that $\Delta\omega$ approximates kv only near cutoff when the phase velocity is very small. At lower frequencies $\Delta\omega/kv$ tends to zero very rapidly as the phase velocity approaches the velocity of light.

It is a little more difficult to carry out the calculation for ΔY for a fixed X since this difference becomes very large near cutoff. Away from cutoff, i.e., for low frequencies, a similar procedure can be used to obtain

$$\Delta Y = -\frac{F}{\partial F/\partial y} \tag{58}$$

with

$$Y = Y_0.$$

The difference in phase velocity with and without plasma drift is then given by

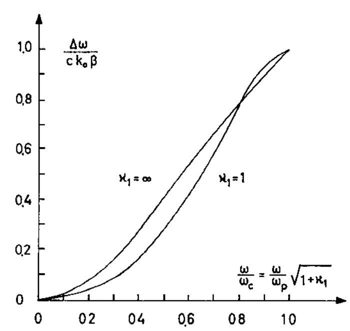

Fig. 4. Frequency shift versus normalized frequency.

$$u - u_0 = \frac{\omega}{k+\Delta k} - \frac{\omega}{k} \cong -\frac{\omega\Delta k}{k_0^2} = -v\frac{X_0\Delta Y}{\beta Y_0^2}. \tag{59}$$

After some algebra we obtain

$$u = u_0 + v\frac{2\kappa_1 X_0^2}{1-2X_0^2-(\kappa_1^2-1)X_0^4}. \tag{60}$$

Much below cutoff, when the phase velocity is high, the drag coefficient multiplying v is negligible and $u \cong u_0$ for all practical purposes.

The results given in (57) and (60) can also be obtained from the perturbation formulas (29) and (36), If we set $\mu_2 = \mu_0$ and $\epsilon_2 = \epsilon_0(1-\omega_p^2/\omega_0^2)$ we get

$$\frac{\Delta k}{v\omega_0} = -\left(\frac{\omega_p}{\omega_0}\right)^2 \cdot \frac{\int_{-\infty}^{0}\left\{(k_0/\omega_0)\epsilon_0 \boldsymbol{E}_0 \cdot \boldsymbol{E}_0^* - 2N_0/c^2\right\}dx}{2\int_{-\infty}^{\infty} N_0 dx} \tag{61}$$

and

$$\frac{v\Delta\omega}{\omega_0} = 2\left(\frac{\omega_p}{\omega_0}\right)^2 \cdot \frac{\int_{-\infty}^{0} \{(k_0/\omega_0)\epsilon_0 E_0 \cdot E_0^* - 2N_0/c^2\} dx}{\int_{-\infty}^{\infty} \{[1 + (\omega_p/\omega_0)^2]\epsilon_0 E_0 \cdot E_0^* + \mu_0 H_0 \cdot H_0^*\} dx}. \quad (62)$$

When one substitutes for the unperturbed fields E_0 and H_0, the preceding expressions reduce to (57) and (60). The unperturbed fields can be obtained from Daly[4] or from (43) by setting $\Lambda = 0$.

V. Discussion

It was shown that the relativistic transformation formulas (1) and (2) for the frequency and wave vector can be used for nonuniform plane waves in composite waveguides provided that all moving media making up the guide are nondispersive and move with the same velocity. When these conditions are not fulfilled one must generally obtain the dispersion relation for the composite guide directly from Maxwell's equations and the constitutive relations. If the dispersion relation is known when all media are stationary, perturbation formulas may be used to obtain the corresponding relation when all or some of the media making up the guide are moving and dispersive.

In the experiments reported in Kerzar and Weisglas,[2] a drifting plasma was enclosed in a glass tube inside a cylindrical waveguide. In the interpretation of the experiments, the authors set $\Delta\omega$ equal to kv for all frequencies. This does not appear to be justified from the above argument. However, since the waves are much slower (k is larger) for the cylindrical waveguide structure of the experiments than for the simple structure considered here, it is conceivable that $\Delta\omega/kv$ may not be too different from unity even at the lower frequencies. In any case the frequency shift for the circular-symmetric mode is too small at these lower frequencies to permit a precise comparison with theory. It would be interesting to repeat these experiments for higher drift velocities.

Appendix

When the conjugate of (18) is premultiplied by $\epsilon e_z \cdot E_0 \times$, use is made of some vector identities and the terms rearranged, one obtains after integration over the guide cross section

$$\int_S \epsilon\{e_z \cdot E_0 \times (\nabla_t \times E_0^*) - jk_0 E_z E_z^*\} dS = j\omega_0 \int_S \mu\epsilon E_0 \times H_0^* \cdot dS - jk_0 \int_S \epsilon E_0 \cdot E_0^* dS. \quad (63)$$

Similarly, one obtains from (19)

$$\int_S \mu\{e_z \cdot H_0 \times (\nabla_t \times H_0^*) - jk_0 H_z H_z^*\} dS = j\omega_0 \int_S \mu\epsilon E_0^* \times H_0 \cdot dS - jk_0 \int \mu H_0 \cdot H_0^* dS. \quad (64)$$

When these equations are added, the right-hand side is equivalent to

$$4j\left\{\omega_0 \int_S \mu\epsilon N_0 dS - k_0 \int_S U_0 dS\right\}.$$

In order to prove (40) we must show that the left-hand sides of (63) and (64) vanish identically.

From vector identities, or by expansion in rectangular components, one can show that

$$\begin{aligned} e_z \cdot E_0 \times (\nabla_t \times E_0^*) &= -E_0 \cdot \nabla_t E_z^* \\ &= E_z^* \nabla_t \cdot E_0 - \nabla_t \cdot (E_z E_0) \\ &= jk_0 E_z E_z^* - \nabla_t \cdot (E_z E_0). \end{aligned}$$

The last identity follows from $\nabla \cdot E_0 = 0$ which implies $\nabla_t \cdot E_0 - jk_0 E_z = 0$. If this result is substituted into (63), the left-hand side becomes

$$-\int_S \epsilon \nabla_t \cdot E_z E_0 dS.$$

Since ϵ is a constant in each subsection S_i of S, it can be taken inside the divergence sign. The integral can therefore be converted into a contour integral which vanishes over a metallic boundary or at infinity. A completely analogous procedure can be used to show that the left-hand side of (64) is also zero.

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